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Differential Equations: Group Lab Project 01

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Cooperative and Competitive Species

Summary: Our group researched the qualities of systems of differential equations that are used to model cooperative and competitive species. We used a combination of qualitative, numerical and analytical techniques to make observations about the general behaviors of the following two pairs of systems:

A $dx/dt = -4x + 3xy$
$dy/dt = -3y + 2xy$

B $dx/dt = 5x - 2x^2 - 4xy$
$dy/dt = 7y - 4xy - 3y^2$

Finding Equilibria: Equilibrium points for both systems were determined by setting each equation equal to zero and then using algebra to find the appropriate values for *x* and for *y* that balanced the equation. For system A there are two equilibria:

$$dx/dt = -4x + 3xy \rightarrow dx/dt = x\,(3y - 4)$$
$$dy/dt = -3y + 2xy \rightarrow dy/dt = y\,(2x - 3)$$

$$3y - 4 = 0 \rightarrow y = 4/3$$
$$2x - 3 = 0 \rightarrow x = 3/2$$

Equilibria at (0, 0) and (4/3, 3/2)

It is known for the above system that setting both variables to equal zero also produces an equilibrium point at (0, 0). This is also true for system B, which has three additional equilibria:

$$dx/dt = 5x - 2x^2 - 4xy \rightarrow dx/dt = x\,(5 - 2x - 4y)$$
$$dy/dt = 7y - 4xy - 3y^2 \rightarrow dy/dt = y\,(7 - 4x - 3y)$$

$$\begin{array}{r} -2\,(5 - 2x - 4y = 0) \\ \underline{7 - 4x - 3y = 0} \\ -3 \qquad + 5y = 0 \\ x = 13/10,\ y = 3/5 \end{array}$$

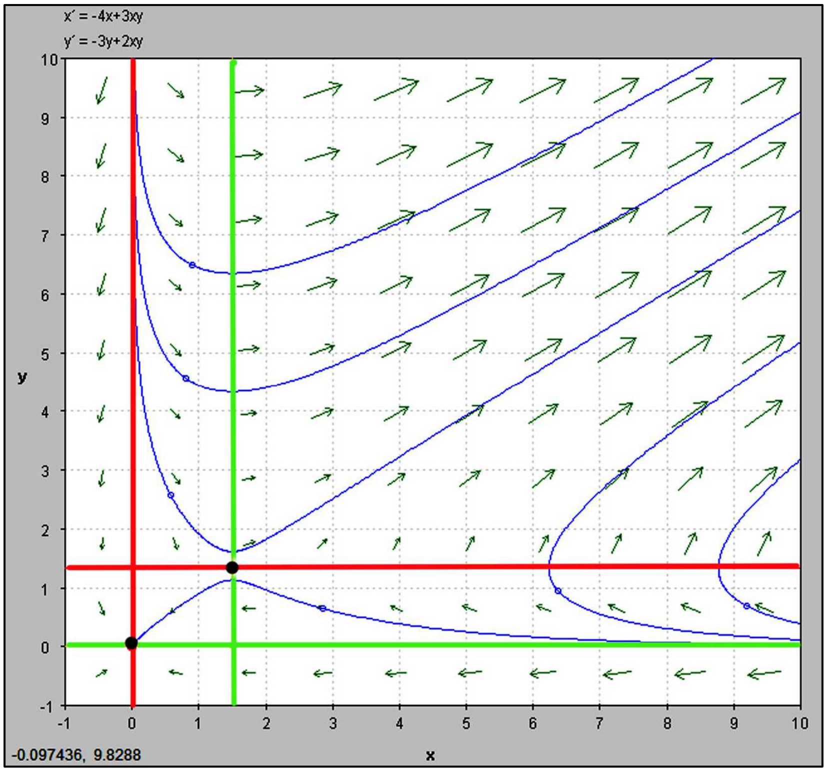
$$y\,(7 - 4x - 3y) = 0,\ x = 0 \rightarrow y\,[7 - 4(0) - 3y] = 0 \rightarrow y = 7/3$$
$$x\,(5 - 2x - 4y) = 0,\ y = 0 \rightarrow x\,[5 - 2x - 4(0)] = 0 \rightarrow x = 2/5$$

Equilibria at (0, 0), (13/10, 3/5), (0, 7/3) and (2/5, 0)

Explanation of Systems: For system A, we found that *x* and *y* share a cooperative relationship as long as neither are zero. When either is equal to zero, the other is effected negatively. Systems of this nature depict two species whose survival depends on a constant relationship with one another, such as bees and flowers: one flower needs to be pollinated by multiple bees to reproduce, and bees gather pollen from multiple flowers for enough food for the colony. The values *x* and *y* must be greater than the equilibrium point of (4/3, 3/2) to see a positive change in growth. This can be attributed to the negative coefficients at the beginning of each equation. If they are equal to the aforementioned point, the change will be zero, resulting in an equal self sustaining population with no growth. If the values of (x, y) decrease less than those mentioned ($0 \le x \le 4/3$, $0 \le y \le 3/2$), the growth rate will become negative causing the population to approach zero.

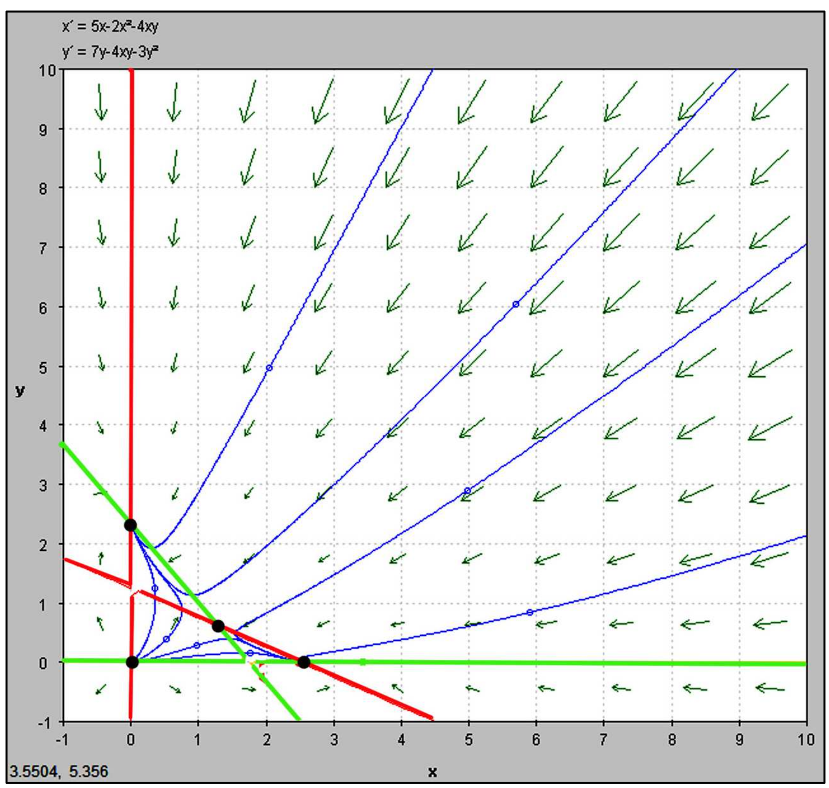
We found system B to be a competitive system. This conclusion was made by observing that when either population comes in contact with the other, the rate at which the population declines increases. Both equations will decline on their own at an exponential rate until the other the population interacts. Only when a population's numbers are at very small (decimal) numbers will there by any positive growth. For the change in *y*, it does seem that the population dies off at a faster rate in this relationship due to its negative coefficient on the $-3y^2$ term. This is the major deciding factor since exponential term's behavior outweighs all others. The same type of behavior also occurs for the change in *x* with the term $-2x^2$.

Behaviors of Solutions:



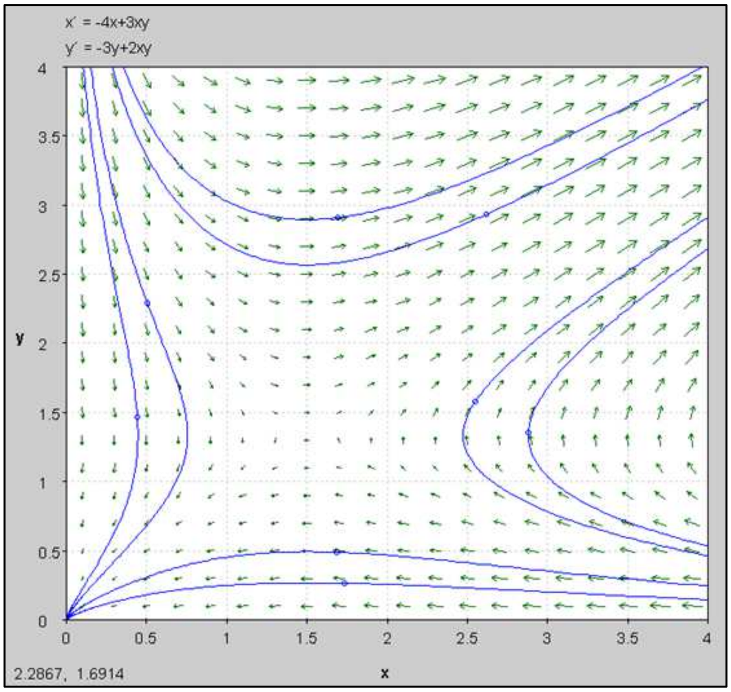
For **System A**, solutions in region I will always head towards positive infinity for both *x* and *y*. Solutions in region II will head towards the bottom right until they approach the *x* nullcline where they then drift away from the equilibrium (4/3, 3/2) towards region I. Solutions in region III will always decline towards the bottom left corner to equilibrium (0, 0). Solutions in region IV will head towards the top left until they approach the y nullcline. They then head away from the source equilibrium (4/3, 3/2) and towards region I.

For **System B**, solutions in region A will head towards the bottom left until reaching the *x* and *y* nullcline where they then move away from the source equilibrium (13/10, 3/5). Solutions with an initial *y* value greater than 1 will go to equilibrium (0, 7/3). When the initial value of *y* is less than 1, the solutions go to the equilibrium point (5/2, 0). Solutions in region II will head towards the bottom right corner until reaching the *y* nullcline where they will then go away from the source equilibrium (13/10, 3/5) towards region III. Solutions in region III will always head in the direction of the bottom right corner towards the equilibrium (0, 0). Solutions in region IV (small bottom triangle) will head towards the top left corner until the x nullcline then head towards region III.



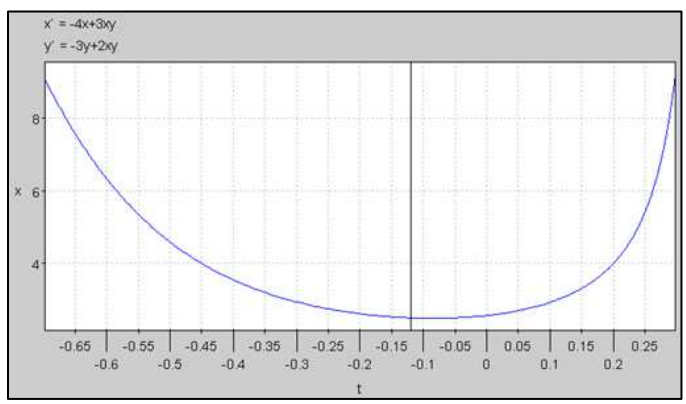
Evolution Scenarios:

System A



This graph shows that for both populations to grow or be sustained the initial population of *x* or *y* must be large. If either value is too small then both populations go extinct.

As shown by this *x* vs. *t* graph, when starting with a large initial value for *x* with a suitably large value for *y,* the size of the population *x* will first decrease as the population of *y* first increases, then the population of *x* will begin to rise with that of *y*.



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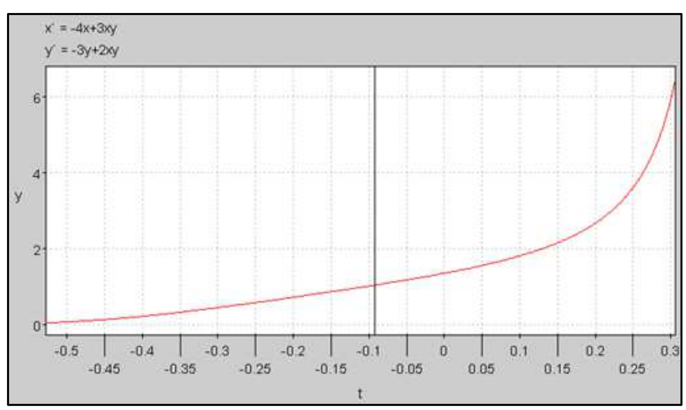
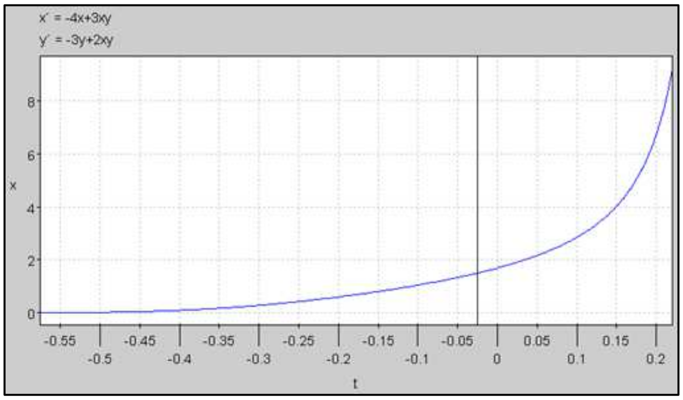
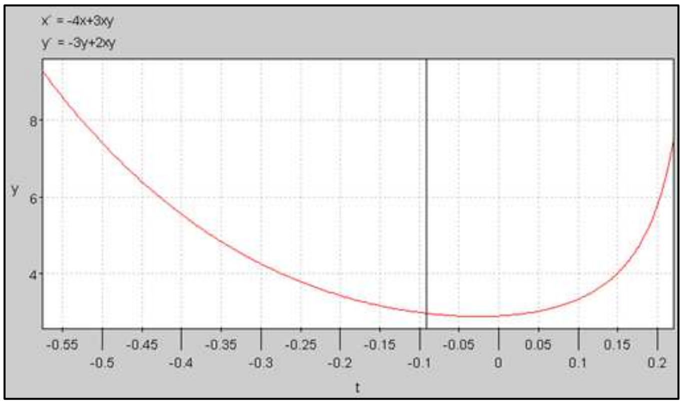


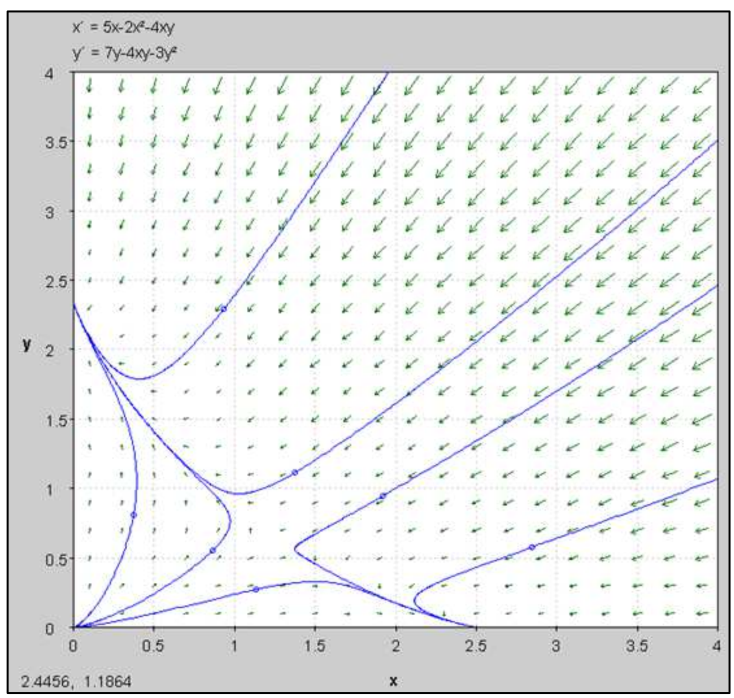


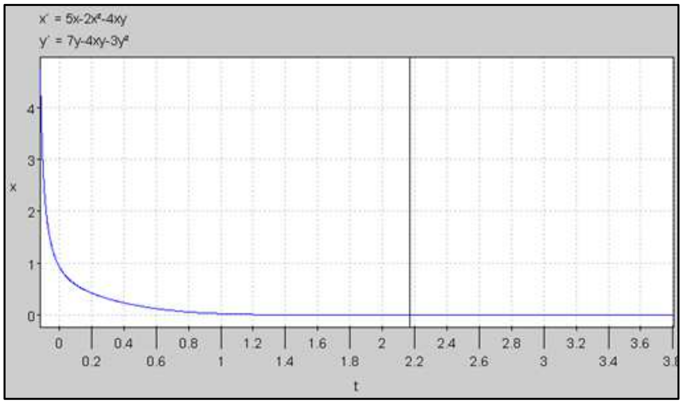


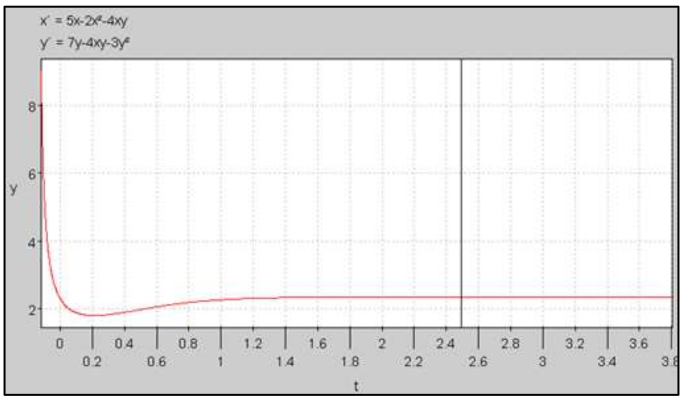
The exact opposite happens for large values of *y*: as *x* increases *y* will decreases initially. Eventually *y* will begin to increase along with *x*. For solutions of this system that are increasing we see both populations go to infinity which is unrealistic for real populations (these equation models do not include carrying capacities).

System B

For very small values of *x* the population of *y* will ultimately increase, and if the value of *y* is great enough the population of *y* will also ultimately increase. However for large values of *x*, the population of *y* ultimately goes extinct while *x* ultimately increases.







For a large value of *y*, the population of *x* continuously decreases till it becomes extinct. On the other hand, *y* initially plunges then recovers, rising to a constant rate of increase in population.

For a large value of *x*, the population of *y* decreases rapidly, temporarily reaching a plateau before it begins declining toward extinction. Meanwhile, the population of *x* dips initially but then rises to a constant rate of increase.