Differential Equations
Project 02: Poles

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Summary: Methods of qualitative investigation are necessary in determining behavioral characteristics of differential equations. These can be used to provide descriptions and explanations for problems without direct solutions, and also as tools in the effective application of mathematical models to real world phenomena.

When applying a Laplace transform to a particular problem, qualitative information may be discerned by observing the characteristics of the transform's poles: the root values which occur in the denominator of the new expression from the original transformed function. This is a qualitative approach which can be used to obtain behavioral explanations with only a small amount of work and to provide important details when analytical approaches are not successful.

Methods: The technique of applying Laplace transforms to differential equations is used to find the poles of eight unique problems, with solutions then being calculated through the required means. Separate graphs for each problem's pole expressions and the solution are presented to find consistencies and general behaviors. A special case of observations is made into the effects of transforming a function, taking its derivative with respect to *s*, and then inverting this final expression to produce a new function.

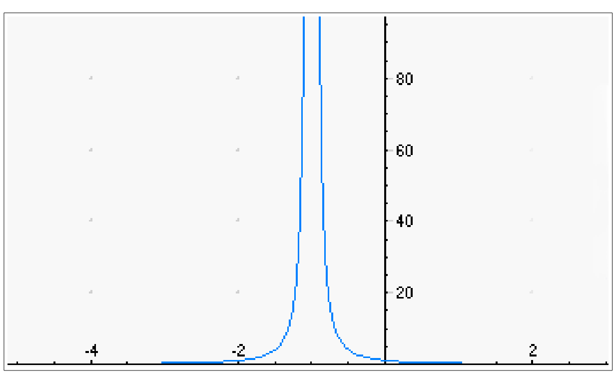
<u>*Equation A:*</u> $\frac{dy}{dt}+y=e^{-t} \qquad y(0)=0$

$$L\left[\frac{dy}{dt}\right]+L[y]=L\left[e^{-t}\right]$$

$$sL[y]-y(0)+L[y]=L\left[e^{-t}\right]$$

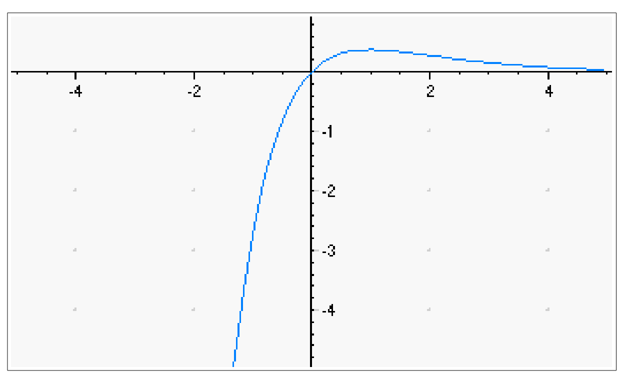
$$(s+1)L[y]=\frac{1}{s+1} \qquad L[y]=\frac{1}{(s+1)^2}$$

$$y(t)=L^{-1}\left[\frac{1}{(s+1)^2}\right]=te^{-t}$$



Graph of $L[y]=\frac{1}{(s+1)^2}$

Double pole at $s=-1$



Graph of solution $y(t)=te^{-t}$

A double pole is observed at $s=-1$. The denominator of the Laplace transform resembles the function $f(t)=te^{-t}$. The solution represents negative exponential growth as $t\to-\infty$ and decays as $t\to\infty$.

<u>*Equation B:*</u> $\frac{dy}{dt}+y=t \qquad y(0)=0$

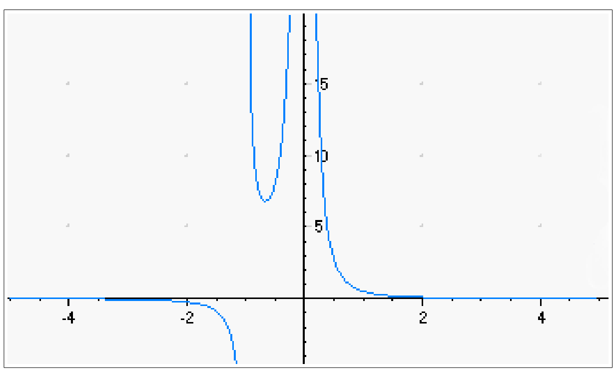
$$L\left[\frac{dy}{dt}\right]+L[y]=L[t]$$

$$sL[y]-y(0)+L[y]=L[t]$$

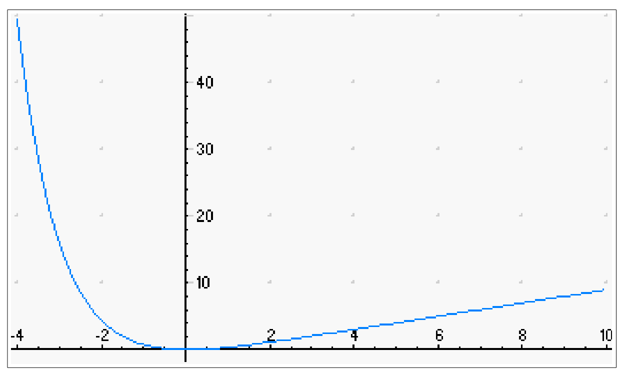
$$(s+1)L[y]=\frac{1}{s^2}$$

$$L[y]=\frac{1}{s^2(s+1)}=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$$

$$y(t)=L^{-1}\left[\frac{1}{s^2(s+1)}\right]=e^{-t}+t-1$$



Graph of $L[y]=\frac{1}{s^2(s+1)}$
Single pole at $s=-1$
Double pole at $s=0$



Graph of solution $y(t)=e^{-t}+t-1$

A single pole is observed at $s=-1$ and a double pole is observed at $s=0$. Splitting the denominator of the Laplace transform into pieces through partial fraction decomposition produces terms that resemble the functions $f(t)=t$, $g(t)=-k$, and $h(t)=e^{-t}$. The $h(t)$ function produces exponential growth and dominates the expression as $t\to-\infty$ while the $f(t)$ function dominates as $t\to\infty$ and produces a line with the slope equal to 1.

<u>Equation C:</u> $\frac{dy}{dt}+y=t^2 \quad y(0)=0$

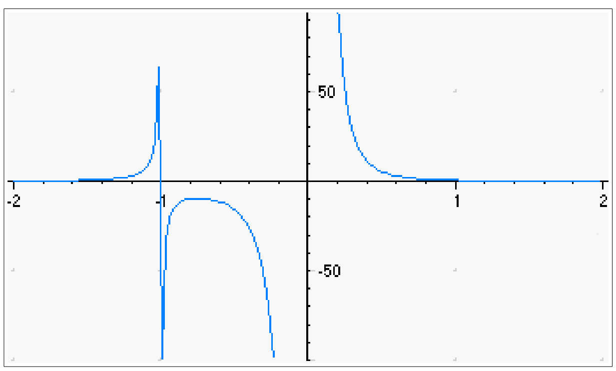
$$L\left[\frac{dy}{dt}\right]+L[y]=L[t^2]$$

$$sL[y]-y(0)+L[y]=L[t^2]$$

$$(s+1)L[y]=\frac{2}{s^3(s+1)}$$

$$L[y]=\frac{1}{s^3(s+1)}=2\left[\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s+1}\right]$$

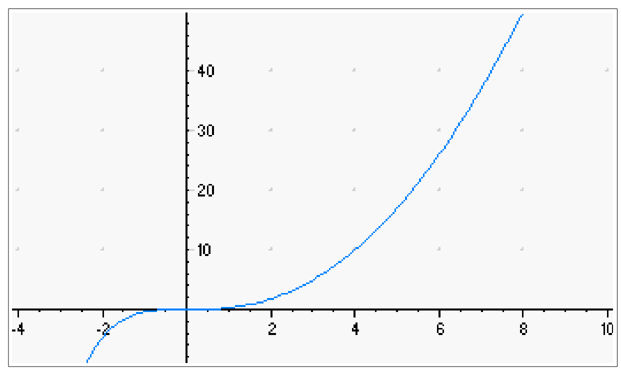
$$y(t)=L^{-1}\left[\frac{2}{s^3(s+1)}\right]=-2e^{-t}+t^2-2\mathrm{t}+2$$



Graph of $L[y]=\frac{1}{s^3(s+1)}$

Single pole at $s=-1$

Triple pole at $s=0$



Graph of solution $y(t)=-2e^{-t}+t^2-2\mathrm{t}+2$

A single pole is observed at $s=-1$ and a triple pole is observed at $s=0$. Splitting the denominator of the Laplace transform into pieces through partial fraction decomposition produces terms that resemble the functions $f(t)=k$, $g(t)=-t$, $h(t)=t^2$, and $q(t)=-e^{-t}$. The $q(t)$ function produces exponential growth and dominates the expression as $t\rightarrow-\infty$ while the $h(t)$ function produces a parabola and dominates as $t\rightarrow\infty$.

Equation D: $\frac{d^2y}{dt^2}+9\mathrm{y}=\sin(3\mathrm{t}) \quad y(0)=0 \quad y'(0)=0$

Calculations to determine poles:

$$L\left[\frac{d^2y}{dt^2}\right]+9\,L[y]=L[\sin(3\mathrm{t})]$$

$$s^2L[y]-s\,y(0)-y'(0)+9\,L[y]=L[\sin(3\mathrm{t})]$$

$$(s^2+9)\,L[y]=\frac{3}{s^2+9}$$

$$L[y]=\frac{3}{(s^2+9)^2}$$

Calculations to determine the solution:

$$y_h(t)=k_1\cos(3\mathrm{t})+k_2\sin(3\mathrm{t})$$

$$y'_h(t)=-3\mathrm{k}_1\sin(3\mathrm{t})+3\mathrm{k}_2\cos(3\mathrm{t})$$

$$y_h(0)=k_1+0=0 \quad k_1=0$$

$$y'_h(0)=0+3\mathrm{k}_2=0 \quad k_2=0$$

$$y_h(t)=0$$

Complexified forcing term: $\frac{d^2y}{dt^2}+9\mathrm{y}=e^{i3t}$

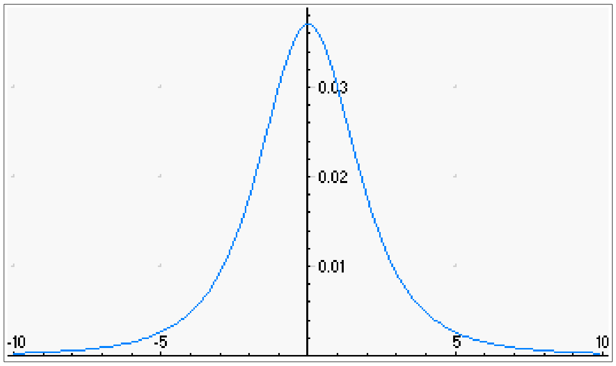
Guessed solution: $y(t)=Ate^{i3t}$

$$(i6Ae^{i3t}-9\mathrm{Ate}^{i3t})+9(Ate^{i3t})=e^{i3t} \quad A=\frac{-i}{6}$$

$$y(t)=\left(\frac{-i}{6}\right)te^{i3t}=\frac{1}{6}tsin(3\mathrm{t})+i\left[-\frac{1}{6}tcos(3\mathrm{t})\right]$$

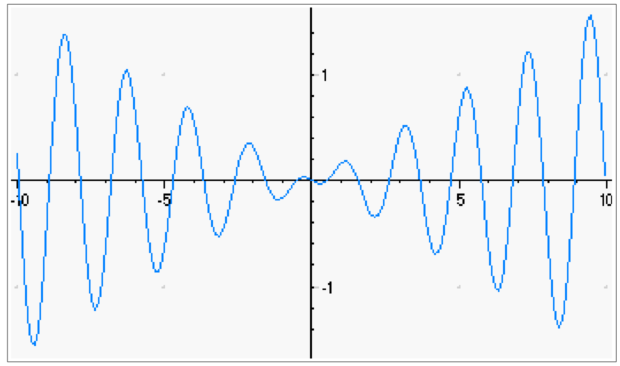
$$y_p(t)=-\frac{1}{6}tcos(3\mathrm{t})$$

$$y(t)=y_h+y_p=-\frac{1}{6}tcos(3\mathrm{t})$$



Graph of $L[y]=\frac{3}{(s^2+9)^2}$

Double pole at $s=\pm i3$



Graph of solution $y(t)=-\frac{1}{6}tcos(3t)$

A double pole is observed at $s=\pm i3$. The solution could not be found by inverting the transform by using known techniques. The period and frequency of the function are $2\pi/3$ and $3/2\pi$, respectively, and were found through the value of the complex number for the pole. A damping term does not exist for this differential equation, which is what causes the solution to increase linearly with an amplitude value of $-t/6$ for all values of t.

Equation E: $\frac{d^2y}{dt^2}+9y=t \qquad y(0)=0 \qquad y'(0)=0$

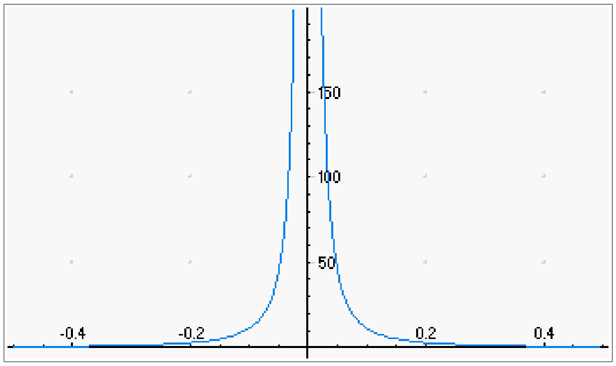
$$L\left[\frac{d^2y}{dt^2}\right]+9L[y]=L[t]$$

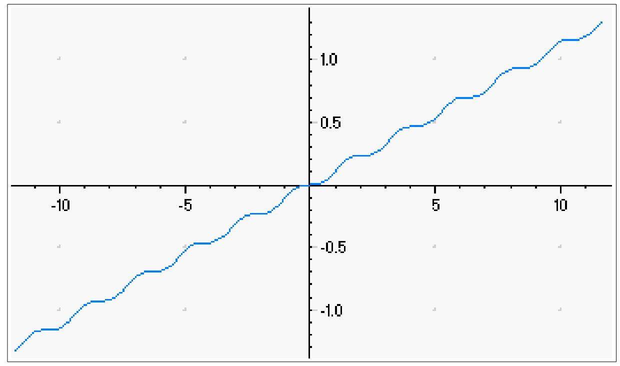
$$s^2L[y]-s\,y(0)-y'(0)+9L[y]=L[t]$$

$$(s^2+9)L[y]=\frac{1}{s^2}$$

$$L[y]=\frac{1}{s^2(s^2+9)}=\frac{1}{9}\left[\frac{1}{s^2}-\frac{1}{s^2+9}\right]$$

$$y(t)=L^{-1}\left[\frac{1}{s^2(s^2+9)}\right]=\frac{1}{9}\left[t-\frac{1}{3}\sin(3t)\right]$$





Graph of $L[y]=\dfrac{1}{s^2(s^2+9)}$

Single pole at $s=\pm i3$

Double pole at $s=0$

Graph of solution $y(t)=\dfrac{1}{9}\left[t-\dfrac{1}{3}\sin(3t)\right]$

A single pole is observed at $s=\pm i3$ and a double pole is observed at $s=0$. Splitting the denominator of the Laplace transform into pieces through partial fraction decomposition produces terms that resemble the functions $f(t)=t$ and $g(t)=-\sin(3t)$. The $f(t)$ function dominates the expression for all values of t and produces a line, while the $g(t)$ function causes small oscillations with amplitude $-1/3$ along this line. The period is $2\pi/3$ and the frequency is $3/2\pi$, which were determined through the value of the complex number for the pole. The damping term is absent from this differential equation.

<u>Equation F:</u> $\dfrac{d^2y}{dt^2}+9y=t^2 \qquad y(0)=0 \qquad y'(0)=0$

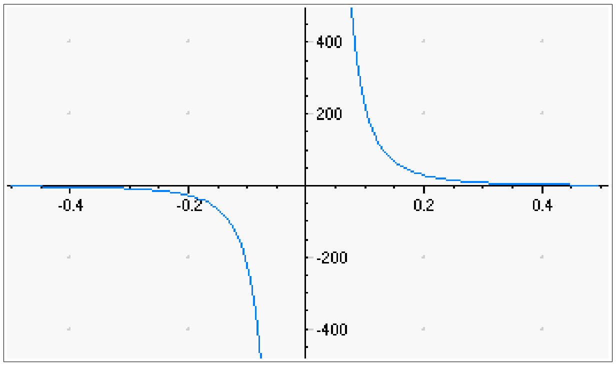
$$L\left[\frac{d^2y}{dt^2}\right]+9\,L[y]=L[t^2]$$

$$s^2L[y]-s\,y(0)-y'(0)+9\,L[y]=L[t^2]$$

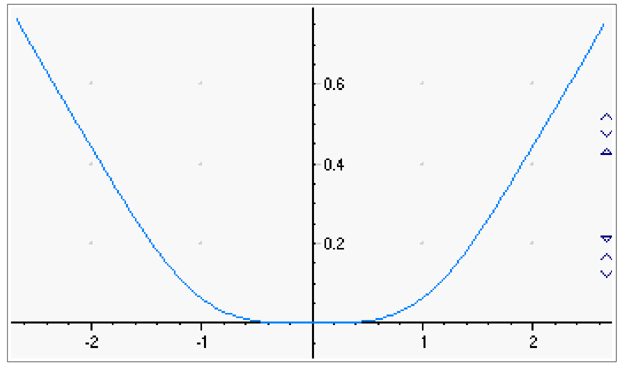
$$(s^2+9)\,L[y]=\frac{2!}{s^3}$$

$$L[y]=\frac{2}{s^3(s^2+9)}=\frac{2}{81}\left[\frac{s}{s^2+9}-\frac{1}{s}\right]+\frac{2}{9}\left(\frac{1}{s^3}\right)$$

$$y(t)=L^{-1}\left[\frac{1}{s^3(s^2+9)}\right]=\frac{2}{81}\cos(3t)+\frac{1}{9}t^2-\frac{2}{81}$$



Graph of $L[y]=\dfrac{2}{s^3(s^2+9)}$
Single pole at $s=\pm i3$
Triple pole at $s=0$



Graph of solution
$y(t)=\dfrac{2}{81}\cos(3t)+\dfrac{1}{9}t^2-\dfrac{2}{81}$

A single pole is observed at $s=\pm i3$ and a triple pole is observed at $s=0$. Splitting the denominator of the Laplace transform into pieces through partial fraction decomposition produces terms that resemble the functions $f(t)=\cos(3t)$, $g(t)=-k$, and $h(t)=t^2$. The $h(t)$ function dominates the expression for all values of t and produces a parabola, while the $f(t)$ function causes miniscule oscillations with amplitude $2/81$ to occur along the parabola. The period is $2\pi/3$ and the frequency is $3/2\pi$, which were determined through the value of the complex number for the pole. The damping term is absent from this differential equation.

Equation G: $\dfrac{d^2y}{dt^2}+3\dfrac{dy}{dt}+2\mathrm{y}=e^{-t}$ $\quad y(0)=0 \quad y'(0)=0$

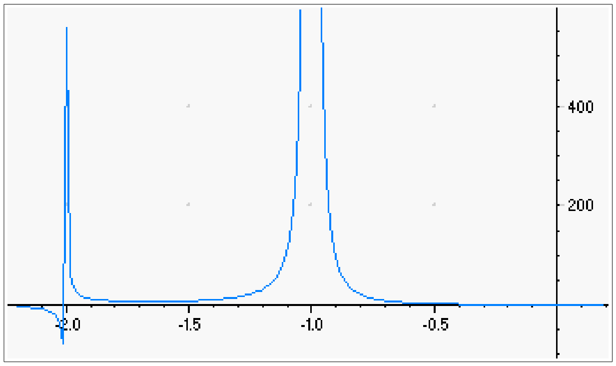
$$L\left[\frac{d^2y}{dt^2}\right]+3L\left[\frac{dy}{dt}\right]+2L[y]=L\left[e^{-t}\right]$$

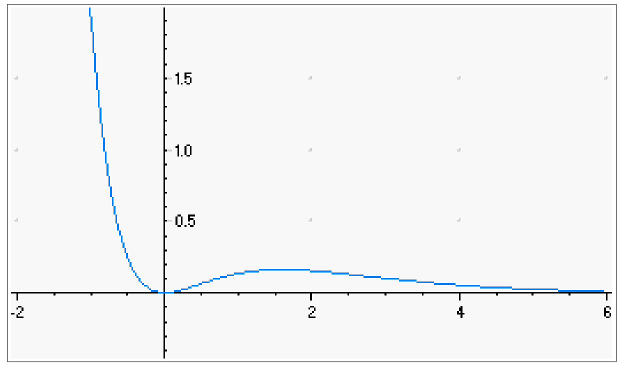
$$s^2L[y]-s\,y(0)-y'(0)+3s\,L[y]-3y(0)+2L[y]=L\left[e^{-t}\right]$$

$$(s^2+3\mathrm{s}+2)L[y]=\frac{1}{s+1}$$

$$L[y]=\frac{1}{(s+1)(s^2+3s+2)}=\frac{1}{s+2}-\frac{1}{s+1}+\frac{1}{(s+1)^2}$$

$$y(t)=L^{-1}\left[\frac{1}{(s+1)(s^2+3s+2)}\right]=e^{-2\mathrm{t}}+(t-1)e^{-t}$$





Graph of $L[y]=\dfrac{1}{(s+1)^2(s+2)}$

Single pole at $s=-2$

Double pole at $s=-1$

Graph of solution

$y(t)=e^{-2t}+(t-1)e^{-t}$

A single pole is observed at $s=-2$ and a double pole is observed at $s=-1$. Splitting the denominator of the Laplace transform into pieces through partial fraction decomposition produces terms that resemble the functions $f(t)=e^{-2t}$, $g(t)=-e^{-t}$, and $h(t)=te^{-t}$. Because the function $f(t)$ moves towards zero more quickly than the other terms, it dominates the expression causing decay as $t \to \infty$.

Equation H: $\dfrac{d^2 y}{dt^2}+2\dfrac{dy}{dt}+y=e^{-t}$ $\quad y(0)=0 \quad y'(0)=0$

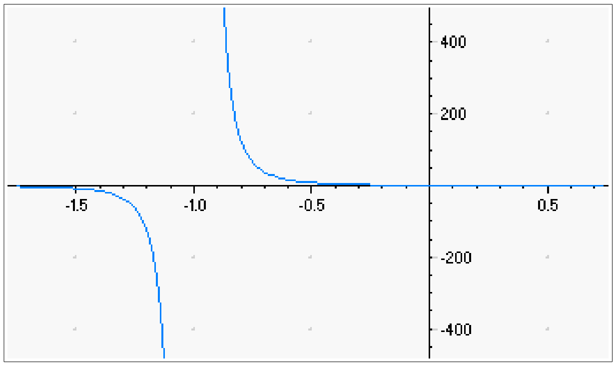
$$L\left[\frac{d^2 y}{dt^2}\right]+2L\left[\frac{dy}{dt}\right]+L[y]=L\left[e^{-t}\right]$$

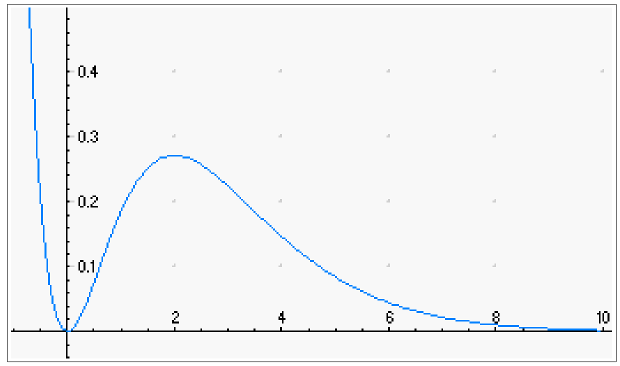
$$s^2 L[y]-s\,y(0)-y'(0)+2\,s\,L[y]-2\,y(0)+L[y]=L\left[e^{-t}\right]$$

$$\left(s^2+2s+1\right)L[y]=\frac{1}{s+1}$$

$$L[y]=\frac{1}{(s+1)\left(s^2+2s+1\right)}=\frac{1}{(s+1)^3}$$

$$y(t)=L^{-1}\left[\frac{1}{(s+1)^3}\right]=\frac{1}{2}t^2 e^{-t}$$





Graph of $L[y]=\dfrac{1}{(s+1)^3}$

Triple pole at $s=-1$

Graph of solution $y(t)=\dfrac{1}{2}t^2e^{-t}$

A triple pole is observed at $s=-1$. The solution will resemble $f(t)=t^2e^{-t}$ with the exponential term dominating the function and causing it to quickly increase as $t\to-\infty$ and then decay toward zero as $t\to\infty$.

<u>*Special Case:*</u> $L^{-1}\left[\dfrac{d}{ds}L[f]\right]$

Because of the unfamiliar nature of the operation, functions producing simple transforms are used to make general observations regarding the behavior. In each case the original function is returned having been multiplied by $-t$.

$f(t)=t$ $\dfrac{d}{ds}(L[t])=\dfrac{d}{ds}\left(\dfrac{1}{s^2}\right)=-2\left(\dfrac{1}{s^3}\right)$ $L^{-1}\left[-2\left(\dfrac{1}{s^3}\right)\right]=-t^2$ $L^{-1}\left[\dfrac{d}{ds}L[t]\right]=-t^2$	$f(t)=t^2$ $\dfrac{d}{ds}(L[t^2])=\dfrac{d}{ds}\left(\dfrac{2!}{s^3}\right)=-6\left(\dfrac{1}{s^4}\right)$ $L^{-1}\left[-6\left(\dfrac{1}{s^4}\right)\right]=-t^3$ $L^{-1}\left[\dfrac{d}{ds}L[t^2]\right]=-t^3$
$f(t)=e^t t$ $\dfrac{d}{ds}(L[e^t t])=\dfrac{d}{ds}\left(\dfrac{1}{(s-1)^2}\right)=-2\left(\dfrac{1}{(s-1)^3}\right)$ $L^{-1}\left[-2\left(\dfrac{1}{(s-1)^3}\right)\right]=-e^t t^2$ $L^{-1}\left[\dfrac{d}{ds}L[e^t t]\right]=-e^t t^2$	$f(t)=e^t t^2$ $\dfrac{d}{ds}(L[e^t t^2])=\dfrac{d}{ds}\left(\dfrac{2!}{(s-1)^3}\right)=-6\left(\dfrac{1}{(s-1)^4}\right)$ $L^{-1}\left[-6\left(\dfrac{1}{(s-1)^4}\right)\right]=-e^t t^3$ $L^{-1}\left[\dfrac{d}{ds}L[e^t t^2]\right]=-e^t t^3$

Further investigation into this operation is done by using Leibnetz's Rule for Integrals and applying it to the definition of a Laplace transform:

$$\frac{d}{ds}\int_a^b f(s,t)\,dt=\int_a^b \frac{\partial}{\partial s} f(s,t)\,dt$$

This expression is used to compute transforms for a for a number of simple functions, with an example provided below for the function $f(t)=t$. In each case the results for these calculations are consistent with the original operation and its final values.

$$f(t)=t$$

$$\int_0^\infty \frac{\partial}{\partial s} t\,e^{-st}\,dt=-\int_0^\infty t^2 e^{-st}\,dt$$

$u=t^2$	$v=\frac{-1}{s}e^{-st}$
$du=2\text{tdt}$	$dv=e^{-st}\,dt$

$$-\int_0^\infty t^2 e^{-st}\,dt=\left[\frac{1}{s}t^2 e^{-st}\right]_0^\infty-\frac{2}{s}\int_0^\infty te^{-st}\,dt$$

$u=t$	$v=\frac{-1}{s}e^{-st}$
$du=dt$	$dv=e^{-st}\,dt$

$$-\frac{2}{s}\int_0^\infty t\,e^{-st}\,dt=\left[\frac{2}{s^2}t\,e^{-st}\right]_0^\infty-\frac{1}{s^2}\int_0^\infty e^{-st}\,dt$$

$$-\int_0^\infty t^2 e^{-st}\,dt=\left[\frac{t^2}{se^{st}}+\frac{2\text{t}}{s^2e^{st}}+\frac{2}{s^3e^{st}}\right]_0^\infty$$

$$\lim_{b\to\infty}\left[\frac{b^2}{se^{sb}}+\frac{2\text{b}}{s^2e^{sb}}+\frac{2}{s^3e^{sb}}-\frac{0}{s}-\frac{0}{s^2}-\frac{2}{s^3}\right]=-\frac{2}{s^3}\ \text{ if }\ s\ >\ 0$$

$$\frac{d}{ds}L[t]=-\frac{2}{s^3}$$

A simple function which does not use Laplace transforms is proposed which involves simply multiplying the original function by $-t$. That is:

$$L^{-1}\left[\frac{d}{ds}L[f]\right]=f(t)\cdot(-t)$$

It is important to note that each time the original operation is applied to a function, in addition to being multiplied by $-t$, the Laplace transform also gains a multiplicity for the pole value. For example:

$$L[t]=\frac{1}{s^2} \text{ and } L[-t^2]=-\frac{2}{s^3} \qquad L[e^t t]=\frac{1}{(s-1)^2} \text{ and } L[-e^t t^2]=-\frac{2}{(s-1)^3}$$

Conclusion: A large amount of qualitative information can be obtained simply by observing the pole values of the Laplace transform for a given function. In some cases it serves as a quick way to identify general behaviors without having to perform the sometimes unwieldy calculations required in partial fraction decomposition procedures, while in others it allows for important details to be gleaned when a solution for a differential equation cannot be produced.

Regarding the special case operation of applying a transform to a function, taking its derivative with respect to s, and then inverting the new expression from the s-domain back to the t-domain, a consistent pattern was found wherein the function was multiplied by $-t$ and the transform gained a multiplicity for the same pole value. Perhaps a practical application of this operation would be to apply it to more complicated functions with unknown or uncommon transforms as a method used with other mathematical techniques to find their associated expressions in the s-domain.